# Directed sets and inverse limits(The interplay between set theory of the reals and iterated forcing)



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# Directed sets and inverse limits

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**Abstract**

We show that the tree property for directed sets is equivalent to the nontriviality of certain inverse limits.

## 1 Directed sets and cofinal types

First we review the basic facts about cofinal types.

**Definition 1.1** Let $\langle D, \leq_D \rangle, \langle E, \leq_E \rangle$ be directed sets. A function $f: E \to D$ which satisfies

$$\forall d \in D \exists e \in E \forall e' \geq_E e\ [f(e') \geq_D d]$$

is called a *convergent function.* If such a function exists we write $D \leq E$ and say *$E$ is cofinally finer than $D$.* $\leq$ is transitive and is called the *Tukey ordering* on the class of directed sets. A function $g: D \to E$ which satisfies

$$\forall e \in E \exists d \in D \forall d' \in D\ [g(d') \leq_E e \to d' \leq_D d]$$

is called a *Tukey function.*

If there exists a directed set $C$ into which $D$ and $E$ can be embedded cofinally, we say *$D$ is cofinally similar with $E$.* In this case we write $D \equiv E$. $\equiv$ is an equivalence relation, and the eqivalence classes with respect to $\equiv$ are the *cofinal types.*

**Proposition 1.2** *For directed sets $D$ and $E$, the follwing are equivalent.*

(a) $D \equiv E$.
(b) $D \leq E$ *and* $E \leq D$.

So we can regard $\leq$ as an ordering on the class of all cofinal types.

**Definition 1.3** For a directed set $D$,

$$\mathsf{add}(D) \stackrel{\text{def}}{=} \min\{|X| \mid X \subseteq D \text{ unbounded}\},$$
$$\mathsf{cof}(D) \stackrel{\text{def}}{=} \min\{|C| \mid C \subseteq D \text{ cofinal}\}.$$

These are the *additivity* and the *cofinality* of a directed set. We restrict ourselves to directed sets $D$ without maximum, so $\mathsf{add}(D)$ is well-defined.

**Proposition 1.4** *For a directed set $D$ (without maximum),*

$$\aleph_0 \leq \mathsf{add}(D) \leq \mathsf{cof}(D) \leq |D|.$$

*Furthermore, $\mathsf{add}(D)$ is regular and $\mathsf{add}(D) \leq \mathsf{cf}(\mathsf{cof}(D))$. Here $\mathsf{cf}$ is the cofinality of a cardinal, which is the same as the additivity of it.*

**Proposition 1.5** *For directed sets $D$ and $E$, $D \leq E$ implies*

$$\mathsf{add}(D) \geq \mathsf{add}(E) \quad \textit{and} \quad \mathsf{cof}(D) \leq \mathsf{cof}(E).$$

From the above proposition we see that these cardinal functions are invariant under cofinal similarity.

# 2 The width of a directed set

In the following, $\kappa$ is always an infinite regular cardinal. If $P$ is partially ordered set, we use the notation $X_{\leq a} = \{x \in X \mid x \leq a\}$ for $X$ a subset of $P$ and $a \in P$. As usual, for cardinals $\kappa \leq \lambda$, $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$ is ordered by inclusion.

**Definition 2.1** The *width* of a directed set $D$ is defined by

$$\mathsf{wid}(D) \stackrel{\text{def}}{=} \sup\{|X|^+ \mid X \text{ is a thin subset of } D\},$$

where 'a *thin* subset of $D$' means

$$\forall d \in D[|X_{\leq d}| < \mathsf{add}(D)].$$

The reason to consider this cardinal function is to give a characterization of the tree proprety. See [2, Theorem 7.1].

**Example 2.2** The set of singletons $\{\{\alpha\} \mid \alpha < \lambda\}$ is thin in $\mathcal{P}_\kappa\lambda$, so we have $\mathsf{wid}(\mathcal{P}_\kappa\lambda) \geq \lambda^+$. If $\kappa$ is strongly inaccessible, then $\mathcal{P}_\kappa\lambda$ is thin in itself, which shows $\mathsf{wid}(\mathcal{P}_\kappa\lambda) = (\lambda^{<\kappa})^+$.

**Lemma 2.3** *For a directed set $D$ and a cardinal $\lambda \geq \kappa := \mathsf{add}(D)$, the following are equivalent.*

(a) *$D$ has a thin subset of size $\lambda$.*
(b) *$D \geq \mathcal{P}_\kappa\lambda$.*
(c) *There exists an order-preserving function $f\colon D \to \mathcal{P}_\kappa\lambda$ with $f[D]$ cofinal in $\mathcal{P}_\kappa\lambda$.*

**Corollary 2.4** *The width of a directed set depends only on its cofinal type.*

**Lemma 2.5** $\mathsf{add}(D)^+ \leq \mathsf{wid}(D) \leq \mathsf{cof}(D)^+$.

# 3 The tree property for directed sets

In the following definition, if $D$ is an infinite regular cardinal $\kappa$, a '$\kappa$-tree on $\kappa$' coincides with the classical '$\kappa$-tree'. Moreover, an 'arbor' is a generalization of a 'well pruned tree'.

**Definition 3.1 ($\kappa$-tree)** ([1]) Let $D$ denote a directed set. A triple $\langle T, \leq_T, s\rangle$ is said to be a *$\kappa$-tree* on $D$ if the following holds.

1) $\langle T, \leq_T\rangle$ is a partially ordered set.
2) $s\colon T \to D$ is an order preserving surjection.
3) For all $t \in T$, $s \restriction T_{\leq t}\colon T_{\leq t} \xrightarrow{\sim} D_{\leq s(t)}$ (order isomorphism).
4) For all $d \in D$, $|s^{-1}\{d\}| < \kappa$. We call $s^{-1}\{d\}$ the *level* $d$ of $T$.

Note that under conditions 1)2)4), condition 3) is equivalent to 3'):

3') (downwards uniqueness principle) $\forall t \in T \forall d' \leq_D s(t) \exists!\, t' \leq_T t\ [s(t') = d']$.

We write $t \!\downarrow\! d$ for this unique $t'$.

If a $\kappa$-tree $\langle T, \leq_T, s\rangle$ satisfies in addition

5) (upwards access principle) $\forall t \in T \forall d' \geq_D s(t) \exists t' \geq_T t\ [s(t') = d']$,

then it is called a *$\kappa$-arbor* on $D$.

**Definition 3.2 (tree property)** ([1]) Let $\langle D, \leq_D \rangle$ be a directed set and $\langle T, \leq_T, s \rangle$ a $\kappa$-tree on $D$. $f: D \to T$ is said to be a faithful embedding if $f$ is an order embedding and satisfies $s \circ f = \mathrm{id}_D$. If for each $\kappa$-tree $T$ on $D$ there is a faithful embedding from $D$ to $T$, we say that $D$ has the *$\kappa$-tree property.* If $D$ has the $\mathsf{add}(D)$-tree property, we say simply $D$ has the *tree property.*

**Proposition 3.3** ([1]) *Let $D$ be directed set and let $\kappa = \mathsf{add}(D)$. $D$ has the tree property iff for any $\kappa$-arbor on $D$ there is a faithful embedding into it.*

**Proposition 3.4** ([1]) *Let $D$ be directed set and let $\theta < \mathsf{add}(D)$. For any $\theta$-tree $T$ on $D$, the number of faithful embeddings from $D$ into $T$ is less than $\theta$.*

**Proposition 3.5** ([1]) *Let $D$ be directed set and let $\theta$ be a cardinal.*

(1) *If $\theta < \mathsf{add}(D)$ then $D$ has the $\theta$-tree property.*
(2) *If $\theta > \mathsf{add}(D)$ then $D$ does not have the $\theta$-tree property.*

Thus we are interested in the case $\theta = \mathsf{add}(D)$.

**Proposition 3.6** ([2]) *If $E$ has the tree property, $D \leq E$ in the Tukey ordering and $\mathsf{add}(D) = \mathsf{add}(E)$, then $D$ also has the tree property. Thus the tree property is a property about the cofinal type of a directed set.*

**Corollary 3.7** ([1]) *If $D$ has the tree property, then $\mathsf{add}(D)$ has the tree property in the classical sense.*

**Theorem 3.8** ([1]) *For a strongly inaccessible cardinal $\kappa$, the following are equivalent:*

(a) *$\kappa$ is strongly compact.*
(b) *All directed sets $D$ with $\mathsf{add}(D) = \kappa$ have the tree property.*

*Condition* (b) *also holds for $\kappa = \aleph_0$.*

# 4 Inverse limits

Now we give a characterization of the tree property in terms of various inverse systems.

**Theorem 4.1** *Let $D$ be a directed set, and let $\theta$ be a cardinal. The following are equivalent:*

(a) *$D$ has the $\theta$-tree property.*
(b) *For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ of sets satisfying $|A_d| < \theta$ for all $d \in D$, the inverse limit $\varprojlim_{d \in D} A_d$ is nonempty.*
(c) *For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ of groups (respectively of abelian groups or free abelian groups), satisfying $|A_d| < \theta$ for all $d \in D$ and $\exists d_0 \in D \forall d \geq d_0\ [f_{d_0 d} \neq 0]$, the inverse limit $\varprojlim_{d \in D} A_d$ has a nonzero element.*
(d) *For any inverse system $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ of vector spaces, satisfying $\mathsf{dim}(A_d) < \theta$ for all $d \in D$ and $\exists d_0 \in D \forall d \geq d_0\ [f_{d_0 d} \neq 0]$, the inverse limit $\varprojlim_{d \in D} A_d$ has a nonzero element.*

**Proof** (a) $\Rightarrow$ (b) Let $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ be an inverse system of nonempty sets, such that $|A_d| < \theta$ for all $d \in D$. Without loss of generality, we may assume that $\langle A_d \mid d \in D \rangle$ is a disjont family. Put $T := \bigcup_{d \in D} A_d$ and define $s: T \to D$ so that $s^{-1}\{d\} = A_d$ for any $d \in D$. For $t, u \in T$ define the ordering $\leq_T$ on $T$ so that

$$t \leq_T u \iff \text{if } t \in A_d, u \in A_{d'} \text{ then } d \leq_D d' \text{ and } f_{dd'}(u) = t.$$

Then $\langle T, \leq_T, s \rangle$ is a $\theta$-tree on $D$, and $\varprojlim_{d \in D} A_d$ is the set of all faithful embeddings from $D$ into $T$. Hence (a) implies (b).

(b) $\Rightarrow$ (a) Let $\langle T, \leq_T, s \rangle$ be a given $\theta$-tree on $D$. Define $f_{dd'}: s^{-1}\{d'\} \to s^{-1}\{d\}$ so that $f_{dd'}(t) = t \downarrow d$.

Then $\langle s^{-1}\{d\}, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ is an inverse system of nonempty sets, and $\varprojlim_{d\in D} s^{-1}\{d\}$ is the set of all faithful embeddings from $D$ into $T$.

(b) $\Rightarrow$ (c) Let $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ be a given inverse system of groups, and assume that $|A_d| < \theta$ for all $d \in D$ and that there is some $d_0 \in D$ such that $f_{d_0 d} \neq 0$ for all $d \geq d_0$. Put

$$\begin{aligned} B_d &:= f_{dd'}[A_{d_0} \setminus \{0\}] && \text{for } d \geq d_0, \\ g_{dd'} &:= f_{dd'} \restriction B_{d'} && \text{for } d' \geq d \geq d_0. \end{aligned}$$

Then $\langle B_d, g_{dd'} \mid d, d' \in D_{\geq d_0},\ d \leq d' \rangle$ is an inverse system of nonempty sets. By (b), we can pick some $b \in \varprojlim_{d \geq d_0} B_d$. Since $D_{\geq d_0}$ is cofinal in $D$ and $D$ is directed, we can extend this $b$ to a unique $a \in \left(\varprojlim_{d\in D} A_d\right) \setminus \{0\}$.

(c) $\Rightarrow$ (b) Let $\langle A_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ be an inverse system of nonempty sets such that $|A_d| < \theta$ for all $d \in D$. Since (a), and hence (b) is always true for $\theta = \aleph_0$, we may assume $\theta > \aleph_0$. For $d \in D$, let $B_d$ be the free abelian group with generators in $A_d$, i.e.

$$B_d := \{b \in {}^{A_d}\mathbb{Z} \mid b(x) = 0 \text{ for all but finitely many } x \in A_d\}.$$

Let $\mathrm{supt}(b) := \{x \in \mathrm{dom}(b) \mid b(x) \neq 0\}$. We identify $b \in B_d$ with the expression $n_0 x_0 + \cdots + n_k x_k$, where $\{x_0, \ldots, x_k\} \supseteq \mathrm{supt}(b)$ and $b(x) = \sum_{\substack{i \leq k \\ x_i = x}} n_i$ for $x \in A_d$. Clearly $|B_d| < \theta$. For $d \leq d'$ in $D$, put

$$\begin{array}{cccc} g_{dd'}: & B_{d'} & \to & B_d \\ & \uplus & & \uplus \\ & n_0 x_0 + \cdots + n_k x_k & \mapsto & n_0 f_{dd'}(x_0) + \cdots + n_k f_{dd'}(x_k). \end{array}$$

Then $\langle B_d, g_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ is an inverse system of free abelian groups, and $g_{dd'} \neq 0$ for any $d \leq d'$ in $D$. Thus by (c), there is some $b^* \in \left(\varprojlim_{d\in D} B_d\right) \setminus \{0\}$. Since $b^* \neq 0$, there is some $d_0 \in D$ such that $b^*(d_0) \neq 0$ for all $d \geq d_0$. Put

$$\begin{aligned} F_d &:= \mathrm{supt}\ b^*(d) \cap f_{d_0 d}^{-1}[\mathrm{supt}\ b^*(d_0)] && \text{for } d \geq d_0, \\ h_{dd'} &:= f_{dd'} \restriction F_{d'} && \text{for } d' \geq d \geq d_0. \end{aligned}$$

Note that $h_{dd'}[F_{d'}] = F_d$. Now $\langle F_d, h_{dd'} \mid d' \geq d \geq d_0 \rangle$ is an inverse system of nonempty finite sets. Since any directed set has the $\aleph_0$-tree proprety, $\varprojlim_{d \geq d_0} F_d \neq \emptyset$. Take any $a \in \varprojlim_{d \geq d_0} F_d$. There is a unique $a' \in \varprojlim_{d\in D} A_d$ which extends $a$.

(b) $\Rightarrow$ (d) This is similar to the proof of (b) $\Rightarrow$ (c).

(d) $\Rightarrow$ (b) This is similar to the proof of (c) $\Rightarrow$ (b). □

**Corollary 4.2** *If $G$ is the inverse limit of $\langle G_d, f_{dd'} \mid d, d' \in D,\ d \leq d' \rangle$ where each $G_d$ is finite (i.e. $G$ is a profinite group), then $G \neq 0$ iff $\exists d_0 \in D \forall d \geq d_0\ [f_{d_0 d} \neq 0]$.*

# References


[1] O.Esser and R.Hinnion, *Large Cardinals and Ramifiability for Directed Sets*, Math. Log. Quart. (1) **46** (2000), 25–34.

[2] M.Karato, *Cofinal types around $\mathcal{P}_\kappa\lambda$ and the tree property for directed sets*, 数理解析研究所講究録 (Sūri kaiseki kenkyūsho kōkyūroku), **1423** (2005), 53–68.